Matematika SMA - COMBINATORICS (KOMBINATORIK)

Wardaya College Winter Camp Olympiad 2017

1. In concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from the set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

2. One some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag.

   We say that a set of N flags is *diverse* if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

3. 2500 chess kings have to be placed on a 100×100 chessboard so that

   (a) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);
   (b) each row and each column contains exactly 25 kings.

   Find the number of such arrengements. (Two arrangements differing by rotation or symmetry are supposed to be different)

4. Cinsider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those with showed gold now show black and vice versa. The last player who can make a legal move wins.

   (a) Does the game necessarily end?
   (b) Does there exist a winning strategy for the starting player?

5. Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

6. Let $D_1, \ldots, D_n$ be closed discs in the plane. A closed disc is the region limited by a circle, taken jointly with this circle. Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1$ other discs D_i.

7. Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with non-selfintersecting boundary) having k internal right angles.

8. In a mathematical competition in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

9. A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state the lamps there exists a sequence of chnages in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

10. There are n markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the sides are up (but not one of the outermost markers) , remove it and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n-1$ is not divisible by 3.