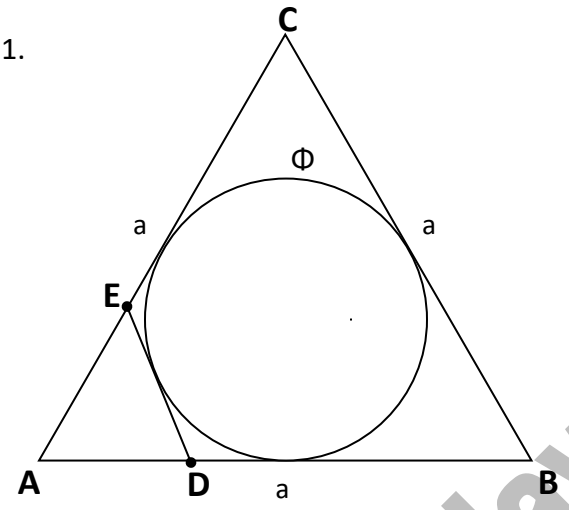
SOLUSI SOAL OLIMPIADE MATEMATIKA

Persiapan Olimpiade Sains Provinsi dan Nasional Tingkat SMA

Departemen Matematika - Wardaya College

MMXVIII-VIII

1.



Dimisalkan; $AB = AC = BC = a$

$BD = p \longrightarrow AD = a - p$

$CE = q \longrightarrow AE = a - q$

$ED + BC = BD + CE$

$ED + a = p + q$

$ED = p + q - a$

Menurut aturan Cosinus:

$ED^2 = AE^2 + AD^2 - 2.AE.AD.\cos 60$

$(p + q - a)^2 = (a - q)^2 + (a - p)^2 - (a - q)(a - p)$

$a = \dfrac{3pq}{p + q}$

Sehingga; $AD = a - p = \dfrac{p(2q - p)}{p + q}$

$AE = a - q = \dfrac{q(2p - q)}{p + q}$

Jadi; $\dfrac{AD}{DB} + \dfrac{AE}{EC} = \dfrac{p(2q - p)}{p(p + q)} + \dfrac{q(2p - q)}{q(p + q)}$

$$\boxed{\dfrac{AD}{DB} + \dfrac{AE}{EC} = \dfrac{p + q}{p + q} = 1}$$

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2. Misalkan titik P (a, a)

$x^2 + y^2 = 1 \longrightarrow a^2 + a^2 = 1$

$$a = -\frac{1}{\sqrt{2}}$$

Maka: $P\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Luas BOC = $\frac{\pi}{4}$

Luas ΔPOC = Luas ΔPOB = $2 \times \frac{1}{2} \times 1 \times \frac{1}{\sqrt{2}}$

$= \frac{1}{2}\sqrt{2}$

$$\boxed{\text{Luas arsiran} = \frac{\pi}{4} + \frac{1}{2}\sqrt{2}}$$

3. C = 3B ; A = 180^0 – 4B

Misal: jari-jari lingkaran luar ABC, R = 1

$a = 2\sin 4B$

$c = 2\sin 3B \longrightarrow c = \frac{10}{3}a$

$10\sin 4B = 3\sin 3B$

$40\sin B\cos B\cos 2B = -12sin^3B + 9\sin B$

$\sin B \neq 0$

Misal: $x = \cos B$

$80x^3 - 12x^2 - 40x + 3 = 0$

$(4x - 3)(20x^2 + 12x - 1) = 0$

$x = \frac{3}{4}$ atau $20x^2 + 12x - 1 = 0$

$(TM)\ 180 - 4B > 0$

$B < 45$

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$x = \cos B = \frac{3}{4}$

$\cos C = \cos 3B = -\frac{9}{16}$

$\cos A = -\cos 4B = \frac{31}{32}$

$$\boxed{\cos A + \cos B + \cos C = \frac{3}{4} + \left(-\frac{9}{16}\right) + \frac{31}{32} = \frac{37}{32}}$$

4. $n^2 - 11n + 63 = k^2$

$4n^2 - 44n + 252 = 4k^2$

$(2n - 11)^2 + 131 = (2k)^2$

$(2k)^2 - (2n - 11)^2 = 131$

$(2k + 2n - 11)(2k - 2n + 11) = 131$

131 adalah bilangan prima

<u>Kasus 1</u>: $2k + 2n - 11 = 131$

$2k - 2n + 11 = 1$ $-$

$4n - 22 = 130 \rightarrow n = 38$

<u>Kasus 2</u>: $2k + 2n - 11 = 1$

$2k - 2n + 11 = 131$ $-$

$4n - 22 = -130 \rightarrow n = -27$

$\boxed{n = 38, -27}$

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5. AM – GM

$$1 + 4x^2 \geq 2\sqrt{4x^2} = 4x$$

$$y \leq \frac{4x^2}{4x} = x$$

$$Dengan\ cara\ yang\ sama: z \leq \frac{4y^2}{4y} = y$$

$$x \leq \frac{4z^2}{4z} = z$$

$$\boxed{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\ dan\ (0,0,0)}$$

6. $\angle AXY = 90^0\ \ (karena\ \ AX^2 + XY^2 = AY^2)$

$$\angle YXC + \angle AXB = 90^0$$

$$\angle XAB + \angle AXB = 90^0$$

$$\angle YXC = \angle XAB$$

$$\therefore \Delta ABX \sim \Delta XCY$$

$$\frac{AB}{4} = \frac{XC}{3} \quad \rightarrow \quad XC = \frac{3}{4}AB$$

$$BX = BC - XC = AB - \frac{3}{4}AB = \frac{1}{4}AB$$

$$PHYTAGORAS:\ AB^2 + \left(\frac{1}{4}AB^2\right) = 4^2$$

$$\boxed{AB = \frac{16}{17}\sqrt{17}}$$

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7. $\left(x\sqrt{x}\right)^x = x^{\frac{3}{2}x}$

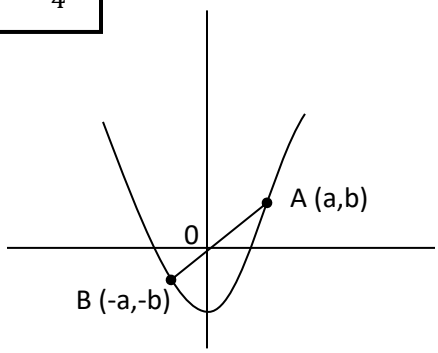
$x^{x\sqrt{x}} = \left(x\sqrt{x}\right)^x$

$x^{x\sqrt{x}} = x^{\frac{3}{2}x}$

$x\sqrt{x} = \frac{3}{2}x$

$\boxed{x = \frac{9}{4}}$ dan $x = 1,\ x = 0$ tidak memenuhi

8.



$Titik\ A$: $b = 2a^2 + 4a - 2$

$Titik\ B$: $-b = (-2a)^2 + (-4a) - 2$

$0 = 4a^2 - 4$

$a = \pm 1$

Untuk a = 1 ; b = 1

Untuk a = -1 ; b = -1

$$\boxed{\overline{AB} = \sqrt{(1+1)^2 + (4+4)^2} = \sqrt{68} = 2\sqrt{17}}$$

9. $Perpotongan\ kedua\ kurva\ tersebut\ adalah$: $x = \pm\frac{\sqrt{3}}{2}\ \ dan\ \ y = \frac{3}{2}$

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$$V = 2 \times 2\pi \int_0^{\frac{3}{2}} \left(2\sqrt{1-x^2} - 1\right) dx$$

$$\boxed{V =}$$

10. $\sqrt{3x^2 - 18x + 52} = 3(x-3)^2 + 25$

$\sqrt{2x^2 - 12x + 162} = 2(x-3)^2 + 144$

$\sqrt{-x^2 + 6x + 280} = -(x-3)^2 + 289$

$Ruas\ Kiri\ \sqrt{25} + \sqrt{144} = 17$

$Ruas\ Kanan\ \sqrt{289} = 17$

$$\boxed{\therefore x = 3}$$

11. Bola-bola yang ada di dalam kotak:

6 bola bernomor ganjil dan 5 bola bernomor genap

Jumlah Ganjil: 1 ganjil dan 5 genap

3 ganjil dan 3 genap

5 ganjil dan 2 genap

$${}_6C_1 \,.\, {}_5C_5 + {}_6C_3 \,.\, {}_5C_3 + {}_6C_5 \,.\, {}_5C_1 = 236$$

$${}_{11}C_6 = 432$$

$$\boxed{\text{PELUANG} = \frac{236}{432} = \frac{118}{231}}$$

12. $x^2 + 2x(y+1) + 3(y+1)^2 + 1 = (x+y+1)^2 + 2(y+1)^2 + 1$

$$\left.\begin{array}{l} Nilai\ minimum\ ketika:\ \ x + y + 1 = 0 \\ \end{array}\right\} \quad x = 0$$

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$$y + 1 = 0 \qquad y = -1$$

$$\boxed{\therefore\ Nilai\ minimum = 1}$$

13. $\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta = 1 \quad ;\ kedua\ ruas\ dikuadratkan\ menjadi:$

$$\frac{9}{25}sin^2\theta + \frac{16}{25}cos^2\theta + \frac{24}{25}\sin\theta\cos\theta = 1$$

$$\frac{9}{25}sin^2\theta + \frac{16}{25}cos^2\theta + \frac{24}{25}\sin\theta\cos\theta = sin^2\theta + cos^2\theta$$

$$\frac{16}{25}sin^2\theta + \frac{9}{25}cos^2\theta - \frac{24}{25}\sin\theta\cos\theta = 0$$

$$\left(\frac{4}{5}\sin\theta - \frac{3}{5}\cos\theta\right)^2 = 0$$

$$\boxed{\tan\theta = \frac{3}{4}}$$

14. Dimisalkan koordinat titik lattice adalah (m, n)

$$x^2 + y^2 = 25$$

$$m^2 + n^2 = 25$$

m dan n adalah bilangan bulat;

$$m^2 \geq 0$$

$$n^2 \in \{0, 1,4,9,16,25\}$$

$$Untuk\ n^2 = 0 \quad \rightarrow m^2 = 25 \quad ; \quad n = 0\ \ dan\ m = \pm 5$$

$$(0,5)\ \&\ (0, -5)$$

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$Untuk\ n^2 = 1 \quad \rightarrow m^2 = 24 \ ; \quad n = \pm 1\ \ dan\ m = \pm\sqrt{24} \quad \therefore\ bukan\ titik\ lattice$

$Untuk\ n^2 = 4 \quad \rightarrow m^2 = 20 \ ; \quad n = \pm 2\ \ dan\ m = \pm\sqrt{20} \quad \therefore\ bukan\ titik\ lattice$

$Untuk\ n^2 = 9 \quad \rightarrow m^2 = 16 \ ; \quad n = \pm 3\ \ dan\ m = \pm 4$

$(3,4); (3,-4); (-3,4); (-3,-4)$

$Untuk\ n^2 = 16 \quad \rightarrow m^2 = 9 \ ; \quad n = \pm 4\ \ dan\ m = \pm 3$

$(4,3); (4,-3); (-4,3); (-4,-3)$

$Untuk\ n^2 = 25 \quad \rightarrow m^2 = 0 \ ; \quad n = \pm 5\ \ dan\ m = 0$

$(5,0)\ \&\ (-5,0)$

$\boxed{\therefore\ Ada\ 12\ titik\ lattice}$

15. $\boxed{\mathrm{n} = 48}$

16. $\boxed{\dfrac{\prod_{k=2}^{2010}(k+1)\prod_{k=2}^{2010}(k-1)}{\prod_{k=2}^{2010}(k)\prod_{k=2}^{2010}(k)} = \dfrac{\prod_{k=3}^{2011}(k)\prod_{k=1}^{2009}(k)}{\prod_{k=2}^{2010}(k)\prod_{k=2}^{2010}(k)} = \dfrac{2011}{2.2010} = \dfrac{2011}{4020}}$

17. $\mathrm{a} = 133333$

$a + (a+1) + \cdots + (a + 400000) = 400001a + 200000(400001)$

$\boxed{= (400001)(333333) = 13333533333}$

18. $x^{\log x} + 10log^{\ -\log x} - 1 = 10$

$a = 10 \qquad\qquad a = 1$

$x^{\log x} = 10 \qquad\qquad x^{\log x} = 1$

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$log^2 x = 10$ $\qquad$ $log^2 x = 0$

$\log x = 1$ & $\log x = -1$ $\qquad$ $\log x = 0$

$x = 10$ & $x = \frac{1}{10}$ $\qquad$ $x = 1$

$$\boxed{x_1 \cdot x_2 \cdot x_3 = 10 \cdot \frac{1}{10} \cdot 1 = 1}$$

19. $\boxed{\frac{2}{2} + \frac{3}{4} = \frac{7}{4}}$

20. Misal: $tan^{-1}\frac{1}{5} = x$ $\qquad$ $tan^{-1}\frac{1}{99} = z$

$tan^{-1}\frac{1}{70} = y$

$$\tan 4x = \frac{2\tan 2x}{1 - tan^2 2x}$$

$$= \frac{2\left(\frac{2\tan x}{1 - tan^2 x}\right)}{1 - \left(\frac{2\tan x}{1 - tan^2 x}\right)^2}$$

$$= \frac{4\tan x(1 - tan^2 x)}{(1 - tan^2 x)^2 - 4tan^2 x}$$

$$= \frac{4\left(\frac{1}{5}\right)\left(1 - \frac{1}{25}\right)}{\left(1 - \frac{1}{25}\right)^2 - 4(25)}$$

$$\tan 4x = \frac{120}{119}$$

$$\tan(y - z) = \frac{\tan y - \tan z}{1 + \tan y \cdot \tan z} = \frac{\frac{1}{70} - \frac{1}{99}}{1 + \frac{1}{70} \cdot \frac{1}{99}} = \frac{1}{239}$$

$$\tan(4x - y + z) = \tan[4x - (y - z)]$$

$$= \frac{\tan 4x - \tan(y - z)}{1 + \tan 4x \cdot \tan(y - z)}$$

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$$= \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}}$$

$$= 1$$

21. $x^2 + (7x+5)^2 = 1$

$x^2 + 49x^2 + 70x + 25 - 1 = 0$

$50x^2 + 70x + 24 = 0$

$25x^2 + 35x + 12 = 0$

$(5x+4)(5x+3) = 0$

$x = -\frac{4}{5} \quad dan \quad x = -\frac{3}{5}$

$\left(-\frac{4}{5}, -\frac{3}{5}\right) \qquad \left(-\frac{3}{5}, \frac{4}{5}\right)$

$\overline{AB} = \sqrt{2}$

$AO = BO = 1$

$\boxed{\angle AOB = 90° \quad \text{(Phytagoras)}}$

22. (i) $x^2 - 3x + 1 = 1$

$x(x-3) = 0 \quad \rightarrow \boxed{x = 0 \;\; dan \; x = 3}$

(ii) $x^2 - 3x + 1 = -1$

$(x-1)(x-2) = 0 \quad \rightarrow \quad x = 1 \;\; dan \; x = 2$

$x + 1 \; bilangan \; genap; \;\; yang \; memenuhi \; adalah \; \boxed{x = 1}$

(iii) $x + 1 = 0 \quad \rightarrow \boxed{x = -1}$

23. $\frac{x_1 - a}{b}, \frac{x_2 - a}{b}, \ldots, \frac{x_n - a}{b}$

$\bar{x}' = \frac{\bar{x} - a}{b} \qquad\qquad Jangkauan' = \frac{J}{b}$

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$\bar{x}' = \dfrac{12-a}{b}$ $\qquad$ $3 = \dfrac{6}{b} \;\rightarrow\; b = 2$

$2 = \dfrac{12-a}{2}$

$a = 8$

$\boxed{\therefore\;\; a + b = 8 + 2 = 10}$

24. $x = 2 \quad \rightarrow\; f(2) + 2f(2010) = 2010 \quad |\times 2|$

$x = 2010 \;\rightarrow\; f(2010) + 2f(2) = 2 \quad |\times 1|$

$$\begin{array}{r} 2f(2) + 4f(2010) = 4020 \\ 2f(2) + \;\; f(2010) \;\; = 2 \quad - \\ \hline 3f(2010) = 4018 \end{array}$$

$f(2010) = \frac{4018}{3}$

25. $\left|\bar{a} + \bar{b}\right|^2 = |\bar{a}|^2 + \left|\bar{b}\right|^2 + |\bar{a}|\left|\bar{b}\right| \cos\theta$

$8 = 16 + 4 + 2 \cdot 4 \cdot 2 \cdot \cos\theta$

$\cos\theta = -\frac{3}{4} \rightarrow \sin\theta = \frac{\sqrt{7}}{4}$

$|a \times b| = |\bar{a}|\left|\bar{b}\right| \sin\theta$

$= 4 \cdot 2 \cdot \frac{\sqrt{7}}{4}$

$\boxed{|a \times b| = 2\sqrt{7}}$

26. $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

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$$\int_{-4}^{5} \frac{x^2}{|x|}\,dx = \int_{-4}^{0} \frac{x^2}{-x}\,dx + \int_{0}^{5} \frac{x^2}{x}\,dx$$

$$= \int_{-4}^{0} -x\,dx + \int_{0}^{5} x\,dx$$

$$= -\frac{1}{2}x^2\Big|_{-4}^{0} + \frac{1}{2}x^2\Big|_{0}^{5}$$

$$= -(0-8) + \left(\tfrac{1}{2} \times 25 - 0\right)$$

$$\boxed{= 8 + \frac{25}{2} = \frac{41}{2}}$$

27. $\frac{d}{dx}\sin x = \cos x$

$\frac{d^2}{dx^2}\sin x = -\sin x$

$\frac{d^3}{dx^3}\sin x = -\cos x$

$\frac{d^4}{dx^4}\sin x = \sin x$

Asumsi : 2 . 1 = - sin

2 . 2 = sin

2 . 3 = - sin

2 . 4 = sin

2010 = 2 . 1005 = - sin

28. $a + b = 16 - c$

$a^2 + b^2 = 160 - c^2$

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$2ab = (a+b)^2 - (a^2+b^2)$

$2ab = (16-c)^2 - (160-c)$

$2ab = 2c^2 - 32c + 96$

$ab = c^2 - 16c + 48$

$acb = c^3 - 32c^2 + 48c$

Agar didapat nilai maksimum, maka turunan pertama harus = 0

$3c^2 - 64c + 48 = 0$

$c = \frac{16 \pm 4\sqrt{7}}{3}$

$untuk\ c = \frac{16+4\sqrt{7}}{3}\ maka\ nilai\ abc < 0$

$maka\ diambil\ c = \frac{16-4\sqrt{7}}{3}$

$$\boxed{\therefore abc = \frac{128}{27}\left(7\sqrt{7} - 10\right)}$$

29. $(1+i)^2 = 2i \qquad (1+i)^3 = 2i - 2$

$f(1+i) = 3(1+i)^3 - 2(1+i)^2 + (1+i) - 3$

$f(1+i) = 3(2i-2) - 2(2i) + (1+i) - 3$

$$\boxed{f(1+i) = -8 + 3i}$$

30. $3^{18} - 2^{18} = (3^9 - 2^9)(3^9 + 2^9)$

$= (3^3 - 2^3)(3^6 + 3^3 \cdot 2^3 + 2^6)(3^3 + 2^3)(3^6 - 3^3 \cdot 2^3 + 2^6)$

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